7.1 Area of a Region Between Two Curves

- **Find the area of a region between two curves using integration.**
- **Find the area of a region between intersecting curves using integration.**
- **Describe integration as an accumulation process.**

Area of a Region Between Two Curves

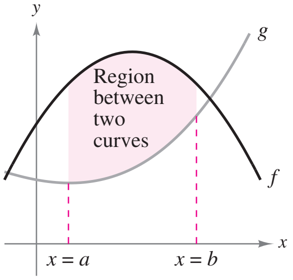
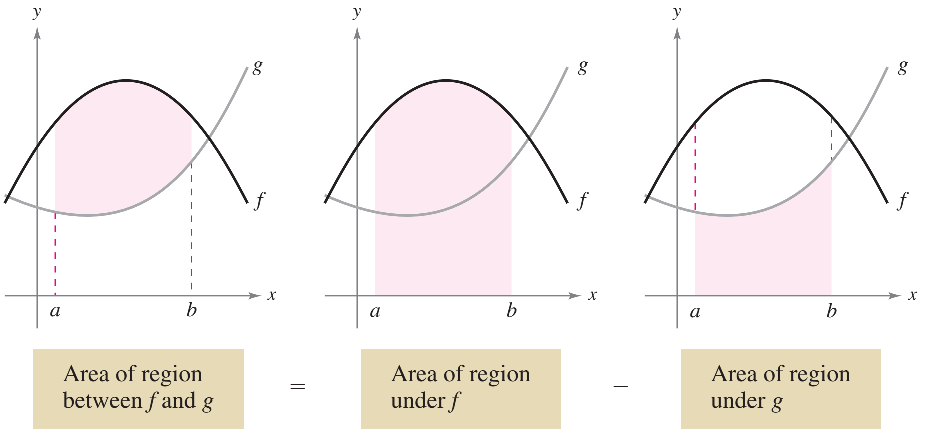


Figure 7.1

With a few modifications, you can extend the application of definite integrals from the area of a region *under* a curve to the area of a region *between* two curves. Consider two functions f and g that are continuous on the interval $[a, b]$. Also, the graphs of both f and g lie above the x-axis, and the graph of g lies below the graph of f, as shown in Figure 7.1. You can geometrically interpret the area of the region between the graphs as the area of the region under the graph of g subtracted from the area of the region under the graph of f, as shown in Figure 7.2.



$$\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

Figure 7.2

To verify the reasonableness of the result shown in Figure 7.2, you can partition the interval $[a, b]$ into n subintervals, each of width Δx. Then, as shown in Figure 7.3, sketch a **representative rectangle** of width Δx and height $f(x_i) - g(x_i)$, where x_i is in the ith subinterval. The area of this representative rectangle is

$$\Delta A_i = (\text{height})(\text{width}) = [f(x_i) - g(x_i)]\Delta x.$$

By adding the areas of the n rectangles and taking the limit as $\|\Delta\| \to 0$ $(n \to \infty)$, you obtain

$$\lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) - g(x_i)]\Delta x.$$

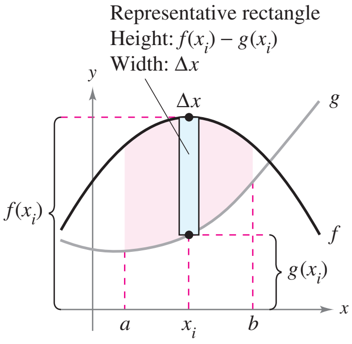


Figure 7.3

REMARK Recall from Section 4.3 that $\|\Delta\|$ is the norm of the partition. In a regular partition, the statements $\|\Delta\| \to 0$ and $n \to \infty$ are equivalent.

Because f and g are continuous on $[a, b]$, $f - g$ is also continuous on $[a, b]$ and the limit exists. So, the area of the region is

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) - g(x_i)]\Delta x = \int_a^b [f(x) - g(x)]\,dx.$$

Area of a Region Between Two Curves

If f and g are continuous on $[a, b]$ and $g(x) \leq f(x)$ for all x in $[a, b]$, then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

In Figure 7.1, the graphs of f and g are shown above the x-axis. This, however, is not necessary. The same integrand $[f(x) - g(x)]$ can be used as long as f and g are continuous and $g(x) \leq f(x)$ for all x in the interval $[a, b]$. This is summarized graphically in Figure 7.4. Notice in Figure 7.4 that the height of a representative rectangle is $f(x) - g(x)$ regardless of the relative position of the x-axis.

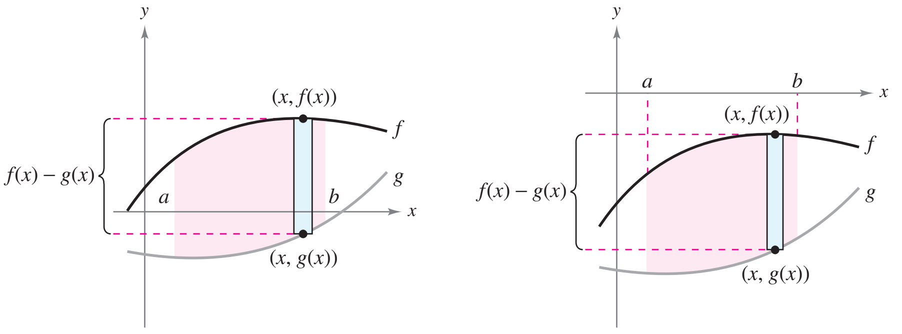


Figure 7.4

Representative rectangles are used throughout this chapter in various applications of integration. A vertical rectangle (of width Δx) implies integration with respect to x, whereas a horizontal rectangle (of width Δy) implies integration with respect to y.

EXAMPLE 1 Finding the Area of a Region Between Two Curves

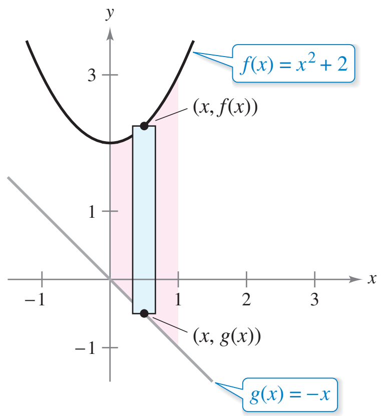
Find the area of the region bounded by the graphs of $y = x^2 + 2$, $y = -x$, $x = 0$, and $x = 1$.

Solution Let $g(x) = -x$ and $f(x) = x^2 + 2$. Then $g(x) \leq f(x)$ for all x in $[0, 1]$, as shown in Figure 7.5. So, the area of the representative rectangle is

$$\begin{aligned}\Delta A &= [f(x) - g(x)]\,\Delta x \\ &= [(x^2 + 2) - (-x)]\,\Delta x\end{aligned}$$

and the area of the region is

$$\begin{aligned}A &= \int_a^b [f(x) - g(x)]\, dx \\ &= \int_0^1 [(x^2 + 2) - (-x)]\, dx \\ &= \left[\frac{x^3}{3} + \frac{x^2}{2} + 2x\right]_0^1 \\ &= \frac{1}{3} + \frac{1}{2} + 2 \\ &= \frac{17}{6}.\end{aligned}$$



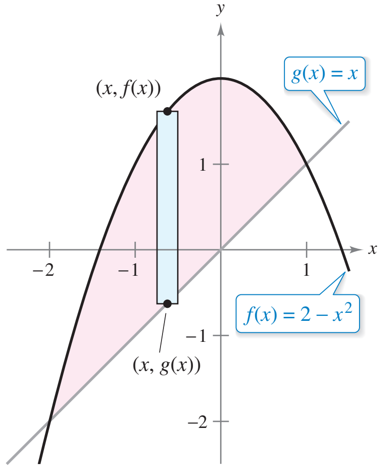
Region bounded by the graph of f, the graph of g, $x = 0$, and $x = 1$
Figure 7.5

Area of a Region Between Intersecting Curves

In Example 1, the graphs of $f(x) = x^2 + 2$ and $g(x) = -x$ do not intersect, and the values of a and b are given explicitly. A more common problem involves the area of a region bounded by two *intersecting* graphs, where the values of a and b must be calculated.

EXAMPLE 2 A Region Lying Between Two Intersecting Graphs

Find the area of the region bounded by the graphs of $f(x) = 2 - x^2$ and $g(x) = x$.



Region bounded by the graph of f and the graph of g
Figure 7.6

Solution In Figure 7.6, notice that the graphs of f and g have two points of intersection. To find the x-coordinates of these points, set $f(x)$ and $g(x)$ equal to each other and solve for x.

$$2 - x^2 = x \qquad \text{Set } f(x) \text{ equal to } g(x).$$
$$-x^2 - x + 2 = 0 \qquad \text{Write in general form.}$$
$$-(x + 2)(x - 1) = 0 \qquad \text{Factor.}$$
$$x = -2 \text{ or } 1 \qquad \text{Solve for } x.$$

So, $a = -2$ and $b = 1$. Because $g(x) \le f(x)$ for all x in the interval $[-2, 1]$, the representative rectangle has an area of

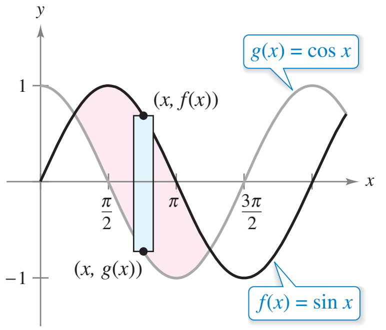
$$\Delta A = [f(x) - g(x)]\,\Delta x = [(2 - x^2) - x]\,\Delta x$$

and the area of the region is

$$A = \int_{-2}^{1} [(2 - x^2) - x]\,dx$$
$$= \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]_{-2}^{1}$$
$$= \frac{9}{2}.$$

EXAMPLE 3 A Region Lying Between Two Intersecting Graphs

The sine and cosine curves intersect infinitely many times, bounding regions of equal areas, as shown in Figure 7.7. Find the area of one of these regions.



One of the regions bounded by the graphs of the sine and cosine functions
Figure 7.7

Solution Let $g(x) = \cos x$ and $f(x) = \sin x$. Then $g(x) \le f(x)$ for all x in the interval corresponding to the shaded region in Figure 7.7. To find the two points of intersection on this interval, set $f(x)$ and $g(x)$ equal to each other and solve for x.

$$\sin x = \cos x \qquad \text{Set } f(x) \text{ equal to } g(x).$$
$$\frac{\sin x}{\cos x} = 1 \qquad \text{Divide each side by } \cos x.$$
$$\tan x = 1 \qquad \text{Trigonometric identity}$$
$$x = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}, \quad 0 \le x \le 2\pi \qquad \text{Solve for } x.$$

So, $a = \pi/4$ and $b = 5\pi/4$. Because $\sin x \ge \cos x$ for all x in the interval $[\pi/4, 5\pi/4]$, the area of the region is

$$A = \int_{\pi/4}^{5\pi/4} [\sin x - \cos x]\,dx$$
$$= \Big[-\cos x - \sin x\Big]_{\pi/4}^{5\pi/4}$$
$$= 2\sqrt{2}.$$

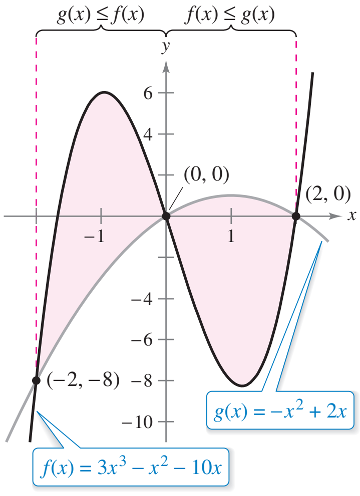
To find the area of the region between two curves that intersect at *more* than two points, first determine all points of intersection. Then check to see which curve is above the other in each interval determined by these points, as shown in Example 4.

EXAMPLE 4 Curves That Intersect at More than Two Points

See LarsonCalculus.com for an interactive version of this type of example.

Find the area of the region between the graphs of

$$f(x) = 3x^3 - x^2 - 10x \quad \text{and} \quad g(x) = -x^2 + 2x.$$



On $[-2, 0]$, $g(x) \le f(x)$, and on $[0, 2]$, $f(x) \le g(x)$.
Figure 7.8

Solution Begin by setting $f(x)$ and $g(x)$ equal to each other and solving for x. This yields the x-values at all points of intersection of the two graphs.

$$3x^3 - x^2 - 10x = -x^2 + 2x \qquad \text{Set } f(x) \text{ equal to } g(x).$$
$$3x^3 - 12x = 0 \qquad \text{Write in general form.}$$
$$3x(x - 2)(x + 2) = 0 \qquad \text{Factor.}$$
$$x = -2, 0, 2 \qquad \text{Solve for } x.$$

So, the two graphs intersect when $x = -2, 0,$ and 2. In Figure 7.8, notice that $g(x) \le f(x)$ on the interval $[-2, 0]$. The two graphs switch at the origin, however, and $f(x) \le g(x)$ on the interval $[0, 2]$. So, you need two integrals—one for the interval $[-2, 0]$ and one for the interval $[0, 2]$.

$$\begin{aligned} A &= \int_{-2}^{0} [f(x) - g(x)]\,dx + \int_{0}^{2} [g(x) - f(x)]\,dx \\ &= \int_{-2}^{0} (3x^3 - 12x)\,dx + \int_{0}^{2} (-3x^3 + 12x)\,dx \\ &= \left[\frac{3x^4}{4} - 6x^2\right]_{-2}^{0} + \left[\frac{-3x^4}{4} + 6x^2\right]_{0}^{2} \\ &= -(12 - 24) + (-12 + 24) \\ &= 24 \end{aligned}$$

REMARK In Example 4, notice that you obtain an incorrect result when you integrate from -2 to 2. Such integration produces

$$\int_{-2}^{2} [f(x) - g(x)]\,dx = \int_{-2}^{2} (3x^3 - 12x)\,dx = 0.$$

When the graph of a function of y is a boundary of a region, it is often convenient to use representative rectangles that are *horizontal* and find the area by integrating with respect to y. In general, to determine the area between two curves, you can use

$$A = \int_{x_1}^{x_2} \underbrace{[(\text{top curve}) - (\text{bottom curve})]}_{\text{in variable } x}\,dx \qquad \text{Vertical rectangles}$$

or

$$A = \int_{y_1}^{y_2} \underbrace{[(\text{right curve}) - (\text{left curve})]}_{\text{in variable } y}\,dy \qquad \text{Horizontal rectangles}$$

where (x_1, y_1) and (x_2, y_2) are either adjacent points of intersection of the two curves involved or points on the specified boundary lines.

EXAMPLE 5 Horizontal Representative Rectangles

Find the area of the region bounded by the graphs of $x = 3 - y^2$ and $x = y + 1$.

Solution Consider

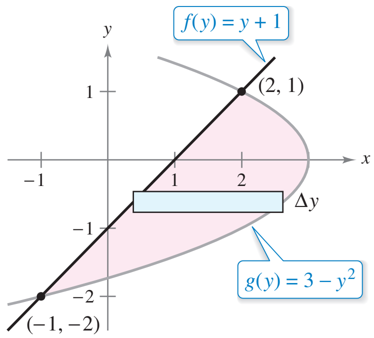
$$g(y) = 3 - y^2 \quad \text{and} \quad f(y) = y + 1.$$

These two curves intersect when $y = -2$ and $y = 1$, as shown in Figure 7.9. Because $f(y) \le g(y)$ on this interval, you have

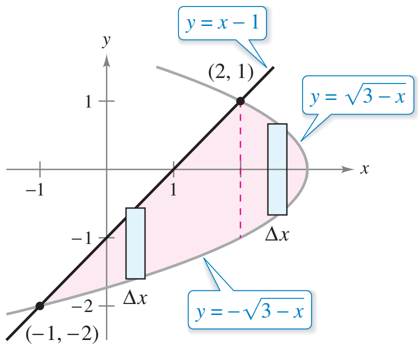
$$\Delta A = [g(y) - f(y)]\,\Delta y = [(3 - y^2) - (y + 1)]\,\Delta y.$$

So, the area is

$$\begin{aligned} A &= \int_{-2}^{1} [(3 - y^2) - (y + 1)]\,dy \\ &= \int_{-2}^{1} (-y^2 - y + 2)\,dy \\ &= \left[\frac{-y^3}{3} - \frac{y^2}{2} + 2y\right]_{-2}^{1} \\ &= \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - 2 - 4\right) \\ &= \frac{9}{2}. \end{aligned}$$



Horizontal rectangles (integration with respect to y)
Figure 7.9



Vertical rectangles (integration with respect to x)
Figure 7.10

In Example 5, notice that by integrating with respect to y, you need only one integral. To integrate with respect to x, you would need two integrals because the upper boundary changes at $x = 2$, as shown in Figure 7.10.

$$\begin{aligned} A &= \int_{-1}^{2} \left[(x - 1) + \sqrt{3 - x}\right] dx + \int_{2}^{3} \left(\sqrt{3 - x} + \sqrt{3 - x}\right) dx \\ &= \int_{-1}^{2} \left[x - 1 + (3 - x)^{1/2}\right] dx + 2\int_{2}^{3} (3 - x)^{1/2}\,dx \\ &= \left[\frac{x^2}{2} - x - \frac{(3 - x)^{3/2}}{3/2}\right]_{-1}^{2} - 2\left[\frac{(3 - x)^{3/2}}{3/2}\right]_{2}^{3} \\ &= \left(2 - 2 - \frac{2}{3}\right) - \left(\frac{1}{2} + 1 - \frac{16}{3}\right) - 2(0) + 2\left(\frac{2}{3}\right) \\ &= \frac{9}{2} \end{aligned}$$

Integration as an Accumulation Process

In this section, the integration formula for the area between two curves was developed by using a rectangle as the *representative element*. For each new application in the remaining sections of this chapter, an appropriate representative element will be constructed using precalculus formulas you already know. Each integration formula will then be obtained by summing or accumulating these representative elements.

Known precalculus formula → Representative element → New integration formula

For example, the area formula in this section was developed as follows.

$A = (\text{height})(\text{width})$ → $\Delta A = [f(x) - g(x)]\,\Delta x$ → $A = \int_a^b [f(x) - g(x)]\,dx$

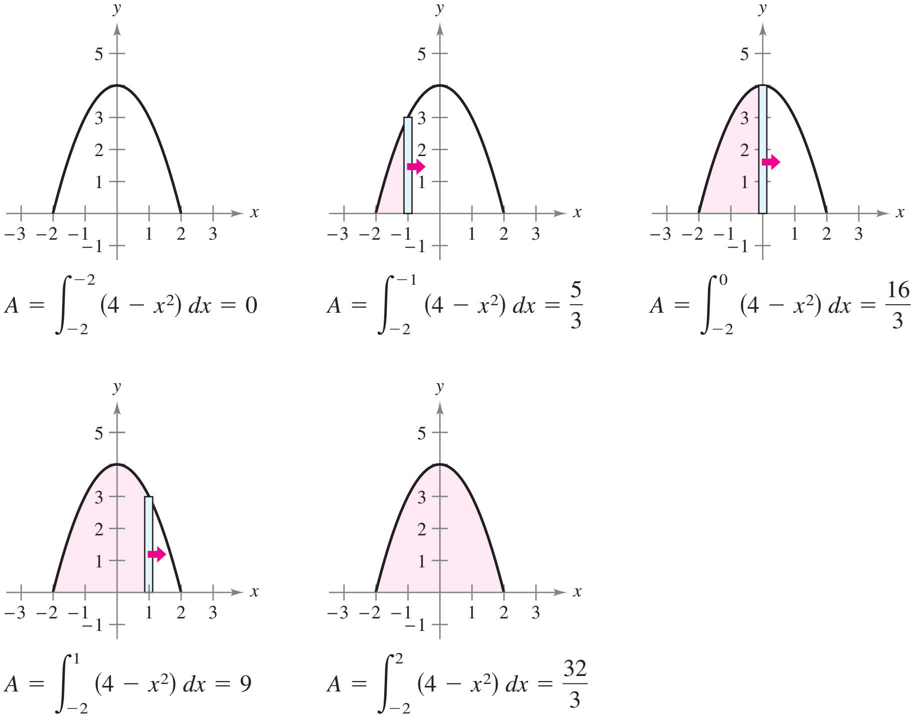
EXAMPLE 6 **Integration as an Accumulation Process**

Find the area of the region bounded by the graph of $y = 4 - x^2$ and the x-axis. Describe the integration as an accumulation process.

Solution The area of the region is

$$A = \int_{-2}^{2} (4 - x^2)\,dx.$$

You can think of the integration as an accumulation of the areas of the rectangles formed as the representative rectangle slides from $x = -2$ to $x = 2$, as shown in Figure 7.11.



$A = \int_{-2}^{-2} (4 - x^2)\,dx = 0$ $A = \int_{-2}^{-1} (4 - x^2)\,dx = \frac{5}{3}$ $A = \int_{-2}^{0} (4 - x^2)\,dx = \frac{16}{3}$

$A = \int_{-2}^{1} (4 - x^2)\,dx = 9$ $A = \int_{-2}^{2} (4 - x^2)\,dx = \frac{32}{3}$

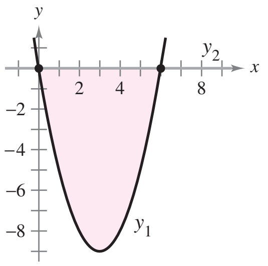
Figure 7.11

7.1 Exercises

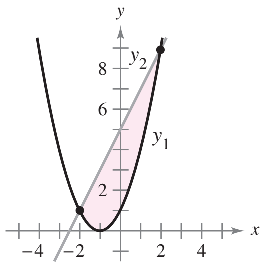
See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Writing a Definite Integral **In Exercises 1–6, set up the definite integral that gives the area of the region.**

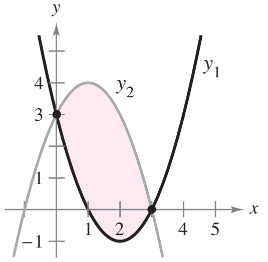
1. $y_1 = x^2 - 6x$
   $y_2 = 0$



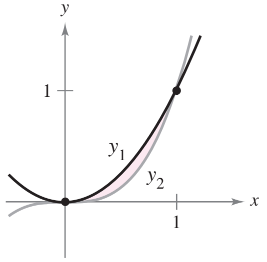
2. $y_1 = x^2 + 2x + 1$
   $y_2 = 2x + 5$



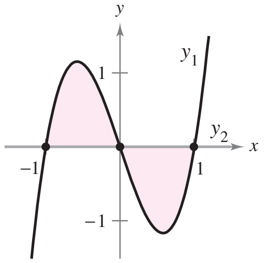
3. $y_1 = x^2 - 4x + 3$
   $y_2 = -x^2 + 2x + 3$



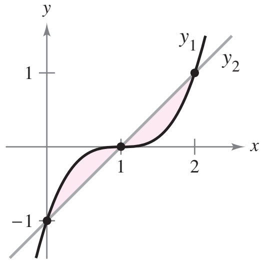
4. $y_1 = x^2$
   $y_2 = x^3$



5. $y_1 = 3(x^3 - x)$
   $y_2 = 0$



6. $y_1 = (x - 1)^3$
   $y_2 = x - 1$



Finding a Region **In Exercises 7–12, the integrand of the definite integral is a difference of two functions. Sketch the graph of each function and shade the region whose area is represented by the integral.**

7. $\displaystyle\int_0^4 \left[(x+1) - \frac{x}{2}\right] dx$

8. $\displaystyle\int_{-1}^1 [(2 - x^2) - x^2]\, dx$

9. $\displaystyle\int_2^3 \left[\left(\frac{x^3}{3} - x\right) - \frac{x}{3}\right] dx$

10. $\displaystyle\int_{-\pi/4}^{\pi/4} (\sec^2 x - \cos x)\, dx$

11. $\displaystyle\int_{-2}^1 [(2 - y) - y^2]\, dy$

12. $\displaystyle\int_0^4 (2\sqrt{y} - y)\, dy$

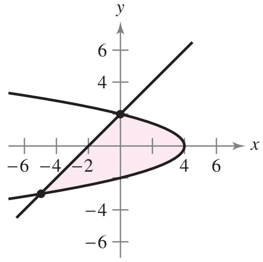
Think About It **In Exercises 13 and 14, determine which value best approximates the area of the region bounded by the graphs of f and g. (Make your selection on the basis of a sketch of the region and not by performing any calculations.)**

13. $f(x) = x + 1, \quad g(x) = (x - 1)^2$
    (a) -2 (b) 2 (c) 10 (d) 4 (e) 8

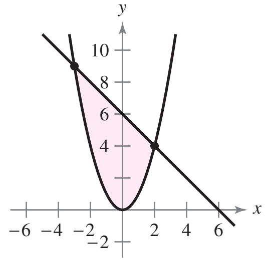
14. $f(x) = 2 - \frac{1}{2}x, \quad g(x) = 2 - \sqrt{x}$
    (a) 1 (b) 6 (c) -3 (d) 3 (e) 4

Comparing Methods **In Exercises 15 and 16, find the area of the region by integrating (a) with respect to x and (b) with respect to y. (c) Compare your results. Which method is simpler? In general, will this method always be simpler than the other one? Why or why not?**

15. $x = 4 - y^2$
    $x = y - 2$



16. $y = x^2$
    $y = 6 - x$



Finding the Area of a Region **In Exercises 17–30, sketch the region bounded by the graphs of the equations and find the area of the region.**

17. $y = x^2 - 1, \quad y = -x + 2, \quad x = 0, \quad x = 1$
18. $y = -x^3 + 2, \quad y = x - 3, \quad x = -1, \quad x = 1$
19. $f(x) = x^2 + 2x, \quad g(x) = x + 2$
20. $y = -x^2 + 3x + 1, \quad y = -x + 1$
21. $y = x, \quad y = 2 - x, \quad y = 0$
22. $y = \dfrac{4}{x^3}, \quad y = 0, \quad x = 1, \quad x = 4$
23. $f(x) = \sqrt{x} + 3, \quad g(x) = \frac{1}{2}x + 3$
24. $f(x) = \sqrt[3]{x - 1}, \quad g(x) = x - 1$
25. $f(y) = y^2, \quad g(y) = y + 2$
26. $f(y) = y(2 - y), \quad g(y) = -y$
27. $f(y) = y^2 + 1, \quad g(y) = 0, \quad y = -1, \quad y = 2$
28. $f(y) = \dfrac{y}{\sqrt{16 - y^2}}, \quad g(y) = 0, \quad y = 3$
29. $f(x) = \dfrac{10}{x}, \quad x = 0, \quad y = 2, \quad y = 10$
30. $g(x) = \dfrac{4}{2 - x}, \quad y = 4, \quad x = 0$

Finding the Area of a Region In Exercises 31–36, (a) use a graphing utility to graph the region bounded by the graphs of the equations, (b) find the area of the region analytically, and (c) use the integration capabilities of the graphing utility to verify your results.

31. $f(x) = x(x^2 - 3x + 3),\quad g(x) = x^2$

32. $y = x^4 - 2x^2,\quad y = 2x^2$

33. $f(x) = x^4 - 4x^2,\quad g(x) = x^2 - 4$

34. $f(x) = x^4 - 9x^2,\quad g(x) = x^3 - 9x$

35. $f(x) = \dfrac{1}{1 + x^2},\quad g(x) = \dfrac{1}{2}x^2$

36. $f(x) = \dfrac{6x}{x^2 + 1},\quad y = 0,\quad 0 \le x \le 3$

Finding the Area of a Region In Exercises 37–42, sketch the region bounded by the graphs of the functions and find the area of the region.

37. $f(x) = \cos x,\quad g(x) = 2 - \cos x,\quad 0 \le x \le 2\pi$

38. $f(x) = \sin x,\quad g(x) = \cos 2x,\quad -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{6}$

39. $f(x) = 2\sin x,\quad g(x) = \tan x,\quad -\dfrac{\pi}{3} \le x \le \dfrac{\pi}{3}$

40. $f(x) = \sec\dfrac{\pi x}{4}\tan\dfrac{\pi x}{4},\quad g(x) = \left(\sqrt{2} - 4\right)x + 4,\quad x = 0$

41. $f(x) = xe^{-x^2},\quad y = 0,\quad 0 \le x \le 1$

42. $f(x) = 2^x,\quad g(x) = \dfrac{3}{2}x + 1$

Finding the Area of a Region In Exercises 43–46, (a) use a graphing utility to graph the region bounded by the graphs of the equations, (b) find the area of the region, and (c) use the integration capabilities of the graphing utility to verify your results.

43. $f(x) = 2\sin x + \sin 2x,\quad y = 0,\quad 0 \le x \le \pi$

44. $f(x) = 2\sin x + \cos 2x,\quad y = 0,\quad 0 < x \le \pi$

45. $f(x) = \dfrac{1}{x^2}e^{1/x},\quad y = 0,\quad 1 \le x \le 3$

46. $g(x) = \dfrac{4\ln x}{x},\quad y = 0,\quad x = 5$

Finding the Area of a Region In Exercises 47–50, (a) use a graphing utility to graph the region bounded by the graphs of the equations, (b) explain why the area of the region is difficult to find by hand, and (c) use the integration capabilities of the graphing utility to approximate the area to four decimal places.

47. $y = \sqrt{\dfrac{x^3}{4 - x}},\quad y = 0,\quad x = 3$

48. $y = \sqrt{x}\,e^x,\quad y = 0,\quad x = 0,\quad x = 1$

49. $y = x^2,\quad y = 4\cos x$

50. $y = x^2,\quad y = \sqrt{3 + x}$

Integration as an Accumulation Process In Exercises 51–54, find the accumulation function F. Then evaluate F at each value of the independent variable and graphically show the area given by each value of F.

51. $F(x) = \displaystyle\int_0^x \left(\frac{1}{2}t + 1\right) dt$ (a) $F(0)$ (b) $F(2)$ (c) $F(6)$

52. $F(x) = \displaystyle\int_0^x \left(\frac{1}{2}t^2 + 2\right) dt$ (a) $F(0)$ (b) $F(4)$ (c) $F(6)$

53. $F(\alpha) = \displaystyle\int_{-1}^{\alpha} \cos\frac{\pi\theta}{2}\,d\theta$ (a) $F(-1)$ (b) $F(0)$ (c) $F\left(\frac{1}{2}\right)$

54. $F(y) = \displaystyle\int_{-1}^{y} 4e^{x/2}\,dx$ (a) $F(-1)$ (b) $F(0)$ (c) $F(4)$

Finding the Area of a Figure In Exercises 55–58, use integration to find the area of the figure having the given vertices.

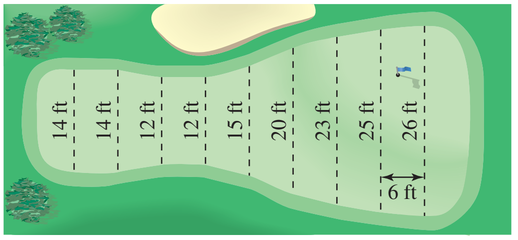
55. $(2, -3), (4, 6), (6, 1)$

56. $(0, 0), (6, 0), (4, 3)$

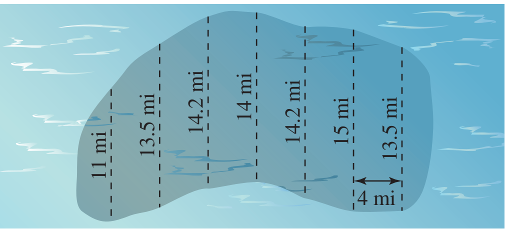
57. $(0, 2), (4, 2), (0, -2), (-4, -2)$

58. $(0, 0), (1, 2), (3, -2), (1, -3)$

59. **Numerical Integration** Estimate the surface area of the golf green using (a) the Trapezoidal Rule and (b) Simpson's Rule.



60. **Numerical Integration** Estimate the surface area of the oil spill using (a) the Trapezoidal Rule and (b) Simpson's Rule.



Using a Tangent Line In Exercises 61–64, set up and evaluate the definite integral that gives the area of the region bounded by the graph of the function and the tangent line to the graph at the given point.

61. $f(x) = x^3,\quad (1, 1)$

62. $y = x^3 - 2x,\quad (-1, 1)$

63. $f(x) = \dfrac{1}{x^2 + 1},\quad \left(1, \dfrac{1}{2}\right)$

64. $y = \dfrac{2}{1 + 4x^2},\quad \left(\dfrac{1}{2}, 1\right)$

WRITING ABOUT CONCEPTS

65. Area Between Curves The graphs of $y = 1 - x^2$ and $y = x^4 - 2x^2 + 1$ intersect at three points. However, the area between the curves *can* be found by a single integral. Explain why this is so, and write an integral for this area.

66. Using Symmetry The area of the region bounded by the graphs of $y = x^3$ and $y = x$ *cannot* be found by the single integral $\int_{-1}^{1} (x^3 - x)\,dx$. Explain why this is so. Use symmetry to write a single integral that does represent the area.

67. Interpreting Integrals Two cars with velocities v_1 and v_2 are tested on a straight track (in meters per second). Consider the following.

$$\int_0^5 [v_1(t) - v_2(t)]\,dt = 10 \qquad \int_0^{10} [v_1(t) - v_2(t)]\,dt = 30$$

$$\int_{20}^{30} [v_1(t) - v_2(t)]\,dt = -5$$

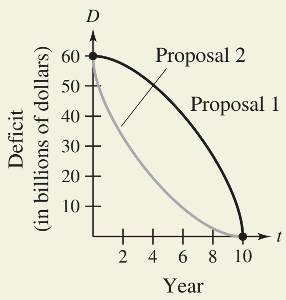
(a) Write a verbal interpretation of each integral.

(b) Is it possible to determine the distance between the two cars when $t = 5$ seconds? Why or why not?

(c) Assume both cars start at the same time and place. Which car is ahead when $t = 10$ seconds? How far ahead is the car?

(d) Suppose Car 1 has velocity v_1 and is ahead of Car 2 by 13 meters when $t = 20$ seconds. How far ahead or behind is Car 1 when $t = 30$ seconds?

68. HOW DO YOU SEE IT? A state legislature is debating two proposals for eliminating the annual budget deficits after 10 years. The rate of decrease of the deficits for each proposal is shown in the figure.



(a) What does the area between the two curves represent?

(b) From the viewpoint of minimizing the cumulative state deficit, which is the better proposal? Explain.

Dividing a Region **In Exercises 69 and 70, find *b* such that the line $y = b$ divides the region bounded by the graphs of the two equations into two regions of equal area.**

69. $y = 9 - x^2, \quad y = 0$ **70.** $y = 9 - |x|, \quad y = 0$

Dividing a Region **In Exercises 71 and 72, find *a* such that the line $x = a$ divides the region bounded by the graphs of the equations into two regions of equal area.**

71. $y = x, \quad y = 4, \quad x = 0$ **72.** $y^2 = 4 - x, \quad x = 0$

Limits and Integrals **In Exercises 73 and 74, evaluate the limit and sketch the graph of the region whose area is represented by the limit.**

73. $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (x_i - x_i^2)\,\Delta x$, where $x_i = \dfrac{i}{n}$ and $\Delta x = \dfrac{1}{n}$

74. $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (4 - x_i^2)\,\Delta x$, where $x_i = -2 + \dfrac{4i}{n}$ and $\Delta x = \dfrac{4}{n}$

Revenue **In Exercises 75 and 76, two models R_1 and R_2 are given for revenue (in billions of dollars) for a large corporation. Both models are estimates of revenues from 2015 through 2020, with $t = 15$ corresponding to 2015. Which model projects the greater revenue? How much more total revenue does that model project over the six-year period?**

75. $R_1 = 7.21 + 0.58t$

$R_2 = 7.21 + 0.45t$

76. $R_1 = 7.21 + 0.26t + 0.02t^2$

$R_2 = 7.21 + 0.1t + 0.01t^2$

77. Lorenz Curve Economists use *Lorenz curves* to illustrate the distribution of income in a country. A Lorenz curve, $y = f(x)$, represents the actual income distribution in the country. In this model, x represents percents of families in the country and y represents percents of total income. The model $y = x$ represents a country in which each family has the same income. The area between these two models, where $0 \le x \le 100$, indicates a country's "income inequality." The table lists percents of income y for selected percents of families x in a country.

x	10	20	30	40	50
y	3.35	6.07	9.17	13.39	19.45

x	60	70	80	90
y	28.03	39.77	55.28	75.12

(a) Use a graphing utility to find a quadratic model for the Lorenz curve.

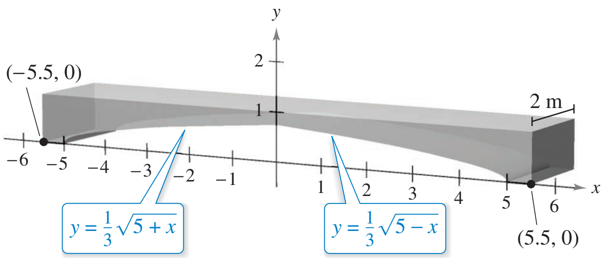
(b) Plot the data and graph the model.

(c) Graph the model $y = x$. How does this model compare with the model in part (a)?

(d) Use the integration capabilities of a graphing utility to approximate the "income inequality."

78. Profit The chief financial officer of a company reports that profits for the past fiscal year were \$15.9 million. The officer predicts that profits for the next 5 years will grow at a continuous annual rate somewhere between $3\frac{1}{2}\%$ and 5%. Estimate the cumulative difference in total profit over the 5 years based on the predicted range of growth rates.

79. Building Design Concrete sections for a new building have the dimensions (in meters) and shape shown in the figure.

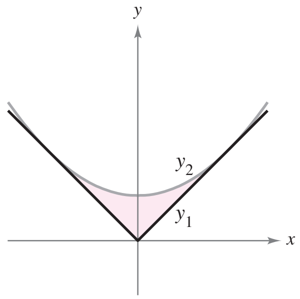


(a) Find the area of the face of the section superimposed on the rectangular coordinate system.

(b) Find the volume of concrete in one of the sections by multiplying the area in part (a) by 2 meters.

(c) One cubic meter of concrete weighs 5000 pounds. Find the weight of the section.

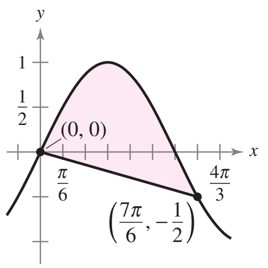
80. Mechanical Design The surface of a machine part is the region between the graphs of $y_1 = |x|$ and $y_2 = 0.08x^2 + k$ (see figure).



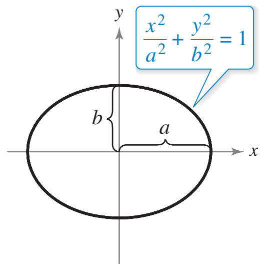
(a) Find k where the parabola is tangent to the graph of y_1.

(b) Find the area of the surface of the machine part.

81. Area Find the area between the graph of $y = \sin x$ and the line segment joining the points $(0, 0)$ and $\left(\frac{7\pi}{6}, -\frac{1}{2}\right)$, as shown in the figure.



82. Area Let $a > 0$ and $b > 0$. Show that the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab (see figure).



True or False? **In Exercises 83–86, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

83. If the area of the region bounded by the graphs of f and g is 1, then the area of the region bounded by the graphs of $h(x) = f(x) + C$ and $k(x) = g(x) + C$ is also 1.

84. If

$$\int_a^b [f(x) - g(x)]\,dx = A$$

then

$$\int_a^b [g(x) - f(x)]\,dx = -A.$$

85. If the graphs of f and g intersect midway between $x = a$ and $x = b$, then

$$\int_a^b [f(x) - g(x)]\,dx = 0.$$

86. The line

$$y = \left(1 - \sqrt[3]{0.5}\right)x$$

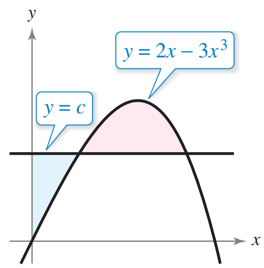
divides the region under the curve

$$f(x) = x(1 - x)$$

on $[0, 1]$ into two regions of equal area.

PUTNAM EXAM CHALLENGE

87. The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant as shown in the figure. Find c so that the areas of the two shaded regions are equal.



This problem was composed by the Committee on the Putnam Prize Competition.

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